Dynamic structure factor of the Calogero-Sutherland model

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We evaluate the dynamic structure factor $S(q,\omega)$ of a one-dimensional quantum Hamiltonian with the inverse-square interaction (Calogero-Sutherland model). For a fixed small q, the structure factor differs from zero in a finite interval of frequencies of the width $\delta\omega \propto q^2/m$. At the borders of this interval $S(q,\omega)$ exhibits power-law singularities with exponents depending on the interaction strength. The singularities are similar in origin to the well-known Fermi-edge singularity in the x-ray absorption spectra of metals.

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Fermi liquid theory proved to be extremely successful in describing interacting fermions [1]. The low-energy excitations of a normal Fermi liquid (FL) are classified the same way as the excitations of a reference system - a non-interacting Fermi gas. A weak residual interaction leads to a finite decay rate $1/\tau$ of FL's quasiparticles. Formally, the rate can be defined as the width of the quasiparticle peak in the spectral function (imaginary part of a single-particle retarded Green function) $A(q,\omega)$. For FL, $A(q,\omega)$ at $T=0$ is a Lorentzian. Its width is easily evaluated in the second order of perturbation theory, which yields $1/\tau \propto \omega^2/\epsilon_F$, where $\omega(q)$ is the quasiparticle energy.

It is well-known, however, that in one dimension (1D) even a weak interaction breaks down the FL description (see [2] for recent reviews). A guide to understanding the properties of interacting 1D systems is provided by the Tomonaga-Luttinger model (TLM) [3], which plays the same role for the concept of the Luttinger liquid [4] as the Fermi gas does for FL. The TLM assumes a strictly linear fermionic dispersion relation. With this assumption, the TLM Hamiltonian can be diagonalized exactly [3], no matter how strong the interactions are. The corresponding elementary excitations are bosons, quanta of the waves of fermionic density. These bosons do not interact, have an infinite lifetime, and propagate without dispersion, $\omega(q) = uq$ (here u is the plasma velocity). Therefore, a measurable quantity, the dynamic structure factor (density-density correlation function) [5]

$$S(q,\omega) = \int dx\, dt\, e^{i(\omega t - qx)} \big\langle \rho(x,t)\rho(0,0)\big\rangle, \qquad (1)$$

has an infinitely sharp peak, $S_{\mathrm{TLM}}(q,\omega) \propto q\delta(\omega - uq)$.

Applications of TLM to the description of "real" 1D fermions rely on the expansion of single-particle energies about the Fermi points $\pm p_F$ [2, 4],

$$\xi_k = \pm v_F k + k^2/2m + \ldots, \quad k = p \mp p_F, \qquad (2)$$

where the upper/lower sign corresponds to the right/left movers (throughout this Letter we use units with $\hbar = 1$). The linear in k term in Eq. (2) is accounted for in TLM. The k^2 term generates interaction between the bosons with the coupling constant $\propto 1/m$ [4], which broadens the peak in $S(q,\omega)$. However, in contrast with FL theory, this broadening is inaccessible by perturbation theory.

This can be seen by considering a special limit of LL, that of non-interacting fermions with spectrum (2). In this case the structure factor differs from zero only if ω lies within a finite interval $\omega_- < \omega < \omega_+$, where $\omega_\pm(q) = v_F q \pm q^2/2m$ for $q < 2p_F$. Within this interval $S(q,\omega)$ is constant, $S = m/q$. Thus, at $q \to 0$ the structure factor indeed approaches the TLM form (with $u = v_F$), but in a very peculiar fashion: the peak in $S(q,\omega)$ at a fixed q has a manifestly non-Lorentzian "rectangular" shape with the width $\delta\omega = q^2/m$. Even this simple result is non-perturbative in the bosonic representation: the first-order in $1/m$ contribution to the boson's self-energy vanishes, while the next one diverges on the mass shell [6].

Recently it was argued [7] that in the presence of interactions between the fermions the shape of the peak in $S(q,\omega)$ remains to be non-Lorentzian. Moreover, even a weak interaction transforms the discontinuities at $\omega = \omega_\pm$ into power-law singularities. Here we approach the problem of finding $S(q,\omega)$ from the perspective of the exactly solvable Calogero-Sutherland model (CSM).

The CSM Hamiltonian reads [8]

$$H = -\sum_{i=1}^{N} \frac{1}{2m}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j} V(x_i - x_j), \qquad (3)$$

where $V(x)$ is a periodic version of $1/x^2$ potential,

$$V(x) = \frac{\lambda(\lambda-1)/m}{(L/\pi)^2 \sin^2(\pi x/L)}\,. \qquad (4)$$

The excitations of CSM can be described in terms of *quasiparticles* and *quasiholes* [8]. Quasiparticles are characterized by velocities v in the range $|v| > u$, and an inertial mass m, the bare mass that enters Eq. (3). Quasiholes have velocities $\bar v$ in the range $|\bar v| < u$, and fractional inertial mass $\bar m = m/\lambda$. The plasma velocity u is given by [8]

$$u = \pi\lambda\rho_0/m, \quad \rho_0 = N/L. \qquad (5)$$

The momentum and energy (relative to the ground state) of an excited state of CSM characterized by a certain set of velocities $\{v_i, \bar{v}_j\}$ read [8]

$$P = \sum_i m v_i - \sum_j \bar{m}\bar{v}_j \tag{6}$$

$$E = \sum_i \frac{1}{2} m\big(v_i^2 - u^2\big) + \sum_j \frac{1}{2}\bar{m}\big(u^2 - \bar{v}_j^2\big). \tag{7}$$

The result of the action of the local density operator

$$\rho_q^\dagger = \int_0^L dx\, e^{iqx}\rho(x), \quad \rho(x) = \frac{1}{L}\sum_{i=1}^N \delta(x - x_i) - \rho_0$$

on CSM's ground state $|0\rangle$ has a remarkable property [9]: $\rho_q^\dagger |0\rangle$ is an eigenstate of CSM in which the velocities of all quasiparticles point in the same direction (positive for $q > 0$). This has a profound effect on the structure factor: $S(q,\omega) \neq 0$ only in a finite interval of frequencies, $\omega_-(q) < \omega < \omega_+(q)$, see Fig. 1(a) (the upper bound ω_+ would be absent in a generic 1D system [7]).

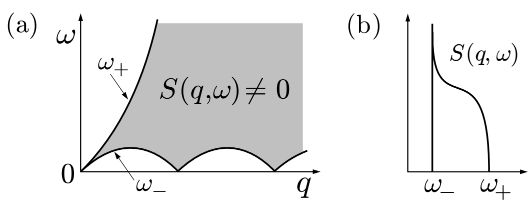

FIG. 1: (a) The structure factor $S(q,\omega)$ differs from zero in a finite interval of frequencies $\omega_- < \omega < \omega_+$. At the borders of this interval $S(q,\omega)$ exhibits power-law singularities, $S \propto |\omega - \omega_\pm|^{\lambda^{\pm 1}-1}$, see Eqs. (21) and (22). The low-energy ($\omega_- \to 0$) sectors correspond to $q = 2\pi\rho_0 I$, where I is an integer. (b) Dependence of $S(q,\omega)$ on ω at a fixed $q \neq 2\pi\rho_0 I$ and for a repulsive interaction ($\lambda > 1$).

From this point on, we consider the rational values of λ only, $\lambda = r/s$, where r and s are co-primes. In this case, the state $|R\rangle = \rho_{q>0}^\dagger |0\rangle$ has exactly s right-moving quasiparticles and exactly r quasiholes [9, 10, 11]. This is the simplest possible excitation that conserves the total inertial mass,

$$sm - r\bar{m} = 0 \quad \text{for} \quad \lambda = r/s. \tag{8}$$

The bounds $\omega_\pm$ are easily found from the momentum and energy conservation,

$$q = P_{|R\rangle}, \quad \omega = E_{|R\rangle}. \tag{9}$$

Indeed, it follows from Eqs. (6)-(9) that

$$q = \sum_{i=1}^s m(v_i - u) + \sum_{j=1}^r \bar{m}(u - \bar{v}_j), \tag{10}$$

$$2(\omega - uq) = \sum_{i=1}^s m(v_i - u)^2 - \sum_{j=1}^r \bar{m}(u - \bar{v}_j)^2. \tag{11}$$

Since $v_i - u \geq 0$ and $u - \bar{v}_j \geq 0$, Eq. (10) implies that for a given $q < 2\pi\rho_0$ the velocities vary in the range

$$\begin{array}{ll} u < v_i < v_0, & v_0 = u + q/m, \\ \bar{v}_0 < \bar{v}_j < u, & \bar{v}_0 = u - q/\bar{m}. \end{array} \tag{12}$$

The upper bound $\omega = \omega_+$ is reached when the velocity of one of the quasiparticles approaches v_0 while all the remaining quasiparticles/holes have velocities close to the plasma velocity u. Eq. (11) then gives

$$\omega_+ - uq = \frac{m}{2}(v_0 - u)^2 = \frac{q^2}{2m}. \tag{13}$$

Similarly, the lower bound ω_- corresponds to the intermediate state $|R\rangle$ in which one of the quasiholes has velocity close to $\bar{v}_0$, while the velocities of all the remaining quasiparticles/holes are close to u, so that

$$\omega_- - uq = -\frac{\bar{m}}{2}(u - \bar{v}_0)^2 = -\frac{q^2}{2\bar{m}} = -\lambda\frac{q^2}{2m}. \tag{14}$$

The width of the region where $S \neq 0$ is then given by [12]

$$\delta\omega = \omega_+ - \omega_- = (\lambda + 1)\frac{q^2}{2m}. \tag{15}$$

Eq. (14) is valid as long as $\omega_- \geq 0$, i.e. for $q \leq 2\pi\rho_0$. At larger q the function $\omega_-(q)$ is given by Eq. (14) with the replacement $q \to q - 2\pi\rho_0 I$, where I is the integer part of $q/2\pi\rho_0$. The corresponding intermediate state $|R\rangle$ has I quasiholes with velocities approaching $-u$, one quasihole with velocity near $\bar{v}_0$ (given by Eq. (12) with the replacement $q \to q - 2\pi\rho_0 I$), and all the remaining quasiparticles/holes moving with the plasma velocity u.

We turn now to the evaluation of the structure factor. In the thermodynamic limit ($N \to \infty$, $\rho_0 = \text{const}$) Eq. (1) can be rewritten as

$$S(q,\omega) = q^2 \int \prod_{i,j} dv_i d\bar{v}_j\, F_{s,r}\, \delta\big(q - P_{|R\rangle}\big)\, \delta\big(\omega - E_{|R\rangle}\big). \tag{16}$$

The form-factor $F_{s,r}$ here is given by

$$F_{s,r} \propto \frac{\prod_{i<i'} |v_i - v_{i'}|^{2\lambda} \prod_{j<j'} |\bar{v}_j - \bar{v}_{j'}|^{2/\lambda}}{\prod_{i,j}(v_i - \bar{v}_j)^2 \prod_i \big(v_i^2 - u^2\big)^{1-\lambda} \prod_j \big(u^2 - \bar{v}_j^2\big)^{1-1/\lambda}}. \tag{17}$$

(note that $\prod dv_i d\bar{v}_j F_{s,r}$ is dimensionless). This expression was conjectured in [10] based on the results of Ref. [13] for $\lambda = 1/2, 2$ [14]; the conjecture was proved in [9] using properties of Jack polynomials. In writing (17), we omitted λ-dependent numerical coefficient [9].

For simplicity, we concentrate on the most interesting limit of small $q \ll \pi\rho_0$. In this limit $v_0 - u, u - \bar{v}_0 \ll u$, see Eq. (12), and one can approximate $v_i^2 - u^2 \approx 2u(v_i - u)$, $u^2 - \bar{v}_j^2 \approx 2u(u - \bar{v}_j)$ in Eq. (17). In view of the

restriction (12) on the velocities, it is convenient to switch in Eq. (16) to the new integration variables

$$x_i = \frac{v_i - u}{v_0 - u}\,, \quad \bar{x}_j = \frac{u - \bar{v}_j}{u - \bar{v}_0}\,, \tag{18}$$

which vary between 0 and 1. In terms of these variables

$$\frac{S(q,\omega)}{S_0(q)} = \int_0^1 \prod dx_i d\bar{x}_j \, F_{s,r}\, \delta\Big(\sum x_i + \sum \bar{x}_j - 1\Big) \tag{19}$$
$$\times\, \delta\bigg(\sum x_i^2 - \lambda \sum \bar{x}_j^2 - (\lambda+1)\frac{\omega - uq}{\delta\omega}\bigg),$$

where $S_0(q) = m/q$ is the structure factor for noninteracting fermions ($\lambda = 1$), and the form-factor is

$$F_{s,r} \propto \frac{\prod_{i<i'} |x_i - x_{i'}|^{2\lambda} \prod_{j<j'} |\bar{x}_j - \bar{x}_{j'}|^{2/\lambda}}{\prod_{i,j}(x_i + \lambda \bar{x}_j)^2 \prod_i x_i^{1-\lambda} \prod_j \bar{x}_j^{1-1/\lambda}}\,. \tag{20}$$

Evaluation of Eq. (19) simplifies considerably when $\omega \to \omega_\pm$. Consider, for example, the limit $\omega - \omega_- \ll \delta\omega$. In this limit the non-vanishing contributions to the integral in (19) come from r sectors of the $(r+s)$-dimensional integration space in which one of $\bar{x}_j$, say $\bar{x}_k$, is close to 1, while all the other variables approach 0. To lowest order in $x_i, \bar{x}_{j\neq k}, 1-\bar{x}_k \ll 1$, the second δ-function in Eq. (19) can be replaced by

$$\delta(\dots) \approx \delta\Big(2\lambda(1-\bar{x}_k) - (\lambda+1)\frac{\omega - \omega_-}{\delta\omega}\Big),$$

and the form-factor $F_{s,r}$ becomes

$$F_{s,r}\Big(\{x_i\},\{\bar{x}_j\}\Big) \approx \lambda^{-2s} F_{s,r-1}\Big(\{x_i\},\{\bar{x}_j\}_{j\neq k}\Big)$$

($F_{s,r-1}$ is given by (20) with $j = k$ excluded). With these approximations, the integrations in (19) are easily carried out and yield

$$\frac{S(q,\omega)}{S_0(q)} \propto \left[\frac{\omega - \omega_-}{\delta\omega}\right]^{1/\lambda - 1}, \quad 0 < \omega - \omega_- \ll \delta\omega. \tag{21}$$

Similarly, at $\omega_+ - \omega \ll \delta\omega$, the non-vanishing contributions to the integral in Eq. (19) come from s sectors where $1 - x_k, x_{i\neq k}, \bar{x}_j \ll 1$, and one finds

$$\frac{S(q,\omega)}{S_0(q)} \propto \left[\frac{\omega_+ - \omega}{\delta\omega}\right]^{\lambda - 1}, \quad 0 < \omega_+ - \omega \ll \delta\omega. \tag{22}$$

It should be emphasized that Eqs. (21) and (22) with q-independent exponents are valid for all $q \neq 2\pi\rho_0 I$, including $q \gtrsim \pi\rho_0$. In addition to $\omega_\pm(q)$, a smooth dependence on q enters Eqs. (21) and (22) via the (omitted) prefactors; these ω-independent prefactors tend to 1 in the limit $\lambda \to 1$. According to Eqs. (21) and (22), the structure factor diverges at $\omega \to \omega_-$ (ω_+) for a repulsive (attractive) interaction, see Fig. 1(b); the divergencies are integrable for all physically meaningful (i.e. positive) values of λ [15]. Note that, formally, Eq. (22) can be obtained by replacing $|\omega - \omega_-| \to |\omega - \omega_+|$ and $\lambda \to 1/\lambda$ in Eq. (21). This correspondence is a manifestation of the particle$\leftrightarrow$hole, $\lambda \leftrightarrow 1/\lambda$ duality of CSM [16].

The power-law singularities in Eqs. (21) and (22) are similar to the familiar edge singularities in the x-ray absorption spectra in metals [17]. Consider, for example, the limit $\omega \to \omega_+$. In this case the action of the operator $\rho_q^\dagger$ on the ground state creates a quasiparticle at $k \approx q$ with energy $\approx \omega_+$. This high-energy quasiparticle interacts with the Fermi sea, resulting in the excitation of multiple low-energy particle-hole pairs, just like the core hole does in the conventional x-ray edge singularity. The proliferation of the low-energy particle-hole pairs leads to the singularity in the response function [7, 18].

This analogy suggests that, just like in the case of the conventional edge singularity, the functional form of the dependence of the structure factor on $\omega - \omega_+$ can be captured by replacing the original model with the properly chosen effective Hamiltonian. For $\omega \to \omega_+$ and $q \ll \pi\rho_0$ it is sufficient to include in the effective Hamiltonian only the *right-moving* single-particle states within two very narrow stripes (subbands) of momenta near $k = q$ and $k = 0$ [7] to allow for both the creation of the high-energy quasiparticle and the low-energy particle-hole pairs. (Recall that for $q < \pi\rho_0$ the velocities of all quasiparticles/holes in the state $\rho_q^\dagger |0\rangle$ are *positive*, see Eq. (12); this would not be the case for a generic interaction [7]).

Upon introducing

$$\psi_r(x) = \sum_{|k|<k_0} \frac{e^{ikx}}{\sqrt{L}}\,\psi_k, \quad \psi_d(x) = \sum_{|k-q|<k_0} \frac{e^{i(k-q)x}}{\sqrt{L}}\,\psi_k,$$

where ψ_k annihilates a right-moving particle with momentum k ($k = 0$ at the Fermi level) and $k_0 \ll q$ is a cutoff, the effective Hamiltonian can be written in the coordinate representation,

$$H_+ = \int dx\, \psi_r^\dagger(-iu\partial_x)\psi_r + \int dx\, \psi_d^\dagger(\omega_+ - iv_0\partial_x)\psi_d$$
$$+ U \int dx\, \rho_d(x)\rho_r(x). \tag{23}$$

Here $\rho_{r,d} = :\psi_{r,d}^\dagger \psi_{r,d}:$, where the colons denote the normal ordering. In Eq. (23) the nonlinearity of spectrum (2) is encoded in the mismatch of the velocities $v_0 - u = q/m$, see Eq. (12). The inter-subband interaction constant U is set by the requirement that the two-particle scattering phase shift $\Theta = U/(v_0 - u)$ for Eq. (23) is equal to that for CSM, $\Theta = (\lambda - 1)\pi$ [8]. This gives

$$U = (v_0 - u)\Theta = (\lambda - 1)\pi q/m, \tag{24}$$

which for $|\lambda - 1| \ll 1$ coincides with $V_0 - V_q$ [7, 19].

In terms of Eq. (23), the structure factor is given by

$$S(q,\omega) = \int dx\, dt\, e^{i\omega t} \big\langle b(x,t) b^\dagger(0,0)\big\rangle, \quad b^\dagger = \psi_d^\dagger \psi_r. \tag{25}$$

Note that the total number of d-particles $N_d = \int dx\, \rho_d(x)$ commutes with H_+ and that the entire d-subband lies above the Fermi level. Hence, as far as the evaluation of Eq. (25) is concerned, H_+ can be further simplified by replacing $\psi_d(x) \to \mathcal{P}\psi_d(x)\mathcal{P}$, where $\mathcal{P}$ is a projector onto states with $N_d = 0, 1$. Obviously, the projected operators satisfy $\rho_d(x)\psi_d(y) = 0$, and $\psi_d(x)\rho_d(y) = \delta(x-y)\psi_d(x)$, which implies that $\big[\rho_d(x), \rho_d(y)\big] = 0$.

We now bosonize the ψ_r-field according to [2]

$$\psi_r(x) = \sqrt{k_0}\, e^{i\varphi(x)}, \quad \big[\varphi(x), \varphi(y)\big] = i\pi\, \mathrm{sgn}(x-y),$$

and apply a unitary transformation [7] with generator $W = (\Theta/2\pi)\int dx\, \rho_d(x)\, \partial_x\varphi$. The transformed Hamiltonian reads

$$\widetilde{H}_+ = e^{iW} H_+ e^{-iW} = H_0 + \delta H, \tag{26}$$

where

$$H_0 = \frac{u}{4\pi}\int dx\, (\partial_x\varphi)^2 + \int dx\, \psi_d^\dagger \big(\omega_+ - iv_0\partial_x\big)\psi_d \tag{27}$$

and $\delta H = (v_0 - u)(\Theta^2/4\pi)\int dx\, \rho_d^2(x)$. It is easy to see that a state with a single d-particle is an eigenstate of δH, $\delta H \psi_d^\dagger(x)|0\rangle \propto (k_0/q)\delta\omega(\lambda-1)^2\psi_d^\dagger(x)|0\rangle$. Thus, when acting in the subspace with $N_d = 0, 1$, the second term in Eq. (26) results merely in a correction to ω_+ in H_0, which for $k_0/q \ll 1$ can be safely neglected, i.e. $\widetilde{H}_+ \approx H_0$.

The same unitary transformation applied to the operator $b^\dagger$ in Eq. (25) yields

$$\widetilde{b}^\dagger(x) = e^{iW} b^\dagger(x)\, e^{-iW} = \sqrt{k_0}\, \psi_d^\dagger(x)\, e^{i\left(1+\Theta/2\pi\right)\varphi(x)}.$$

Evaluation of the correlation function (25) with quadratic Hamiltonian H_0 is now straightforward. The structure factor vanishes identically at $\omega > \omega_+$, while at $\omega < \omega_+$ it is given by Eq. (22). Thus, the outlined simplified description indeed reproduces the exact result (22) with logarithmic accuracy. (The cutoff k_0 would enter Eq. (22) via a factor q/k_0 in the square brackets.) Note that the exponent in Eq. (22) is independent of q. This independence is a direct consequence of the fact that the phase shift $\Theta = \mathrm{const}$ for inverse-square interaction.

Similar reasoning can be applied to the calculation of $S(q,\omega)$ at $\omega \to \omega_-$. In this case the d-subband lies near $k = -q$ well below the Fermi level and carries at most a single hole [7]. After the particle-hole transformation $\psi_{r,d} \to \psi_{r,d}^\dagger$ the corresponding effective Hamiltonian H_- takes the form of Eq. (23) with replacements $\omega_+ \to \omega_-$ and $v_0 \to \bar{v}_0$. Evaluation of $S(q,\omega)$ [which is again given by Eq. (25)] proceeds similar to above and yields Eq. (21).

To conclude, in this Letter we evaluated the dynamic structure factor $S(q,\omega)$ of the Calogero-Sutherland model. Besides being of a fundamental interest for it's own sake, the detailed knowledge of the structure factor for interacting fermions with nonlinear dispersion is important for the description of a variety of effects associated with the particle-hole asymmetry.

We found that $S(q,\omega)$ differs from zero in a finite interval of frequencies. At the borders of this interval $S(q,\omega)$ exhibits power-law singularities, analogous to the edge singularities in the x-ray absorption spectra of metals. Exploiting this analogy, we showed that the exact results (21) and (22) can be reproduced with logarithmic accuracy by replacing the original model with simple effective Hamiltonians. Remarkably, the analogy with the x-ray singularity, previously established for weak interaction only [7], is useful even when interactions are strong. Moreover, similar ideas can be applied to the evaluation of single-particle correlation functions [20].

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[1] P. Nozières and D. Pines, *The Theory of Quantum Liquids* (Perseus Books, Cambridge, MA, 1999).

[2] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, 2004); A.O. Gogolin, A.A. Nersesyan, and A.M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, 1998); D.L. Maslov, in *Nanophysics: Coherence and Transport*, eds. H. Bouchiat *et al.* (Elsevier, 2005), pp. 1-108.

[3] S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950); J.M. Luttinger, J. Math. Phys. **4**, 1154 (1963); D.C. Mattis and E.H. Lieb, J. Math. Phys. **6**, 304 (1965).

[4] F.D.M. Haldane, J. Phys. C **14**, 2585 (1981).

[5] By the fluctuation-dissipation theorem, $S(q,\omega) = 2\,\mathrm{Im}\,\chi(q,\omega)$ at $\omega > 0$ and $T = 0$, where χ is the corresponding retarded correlation function, see, e.g., [1].

[6] K.V. Samokhin, J. Phys. Condens. Matter **10**, L533 (1998).

[7] M. Pustilnik *et al.*, Phys. Rev. Lett. **96**, 196405 (2006).

[8] B. Sutherland, *Beatiful Models* (World Scientific, 2004).

[9] Z.N.C. Ha, Phys. Rev. Lett. **73**, 1574 (1994); **74**, 620(E) (1995); Nucl. Phys. B **435**, 604 (1995).

[10] F.D.M. Haldane, in *Proceedings of the International Colloquium on Modern Field Theory*, eds. G. Mandal *et al.*, (World Scientific, 1995).

[11] M.R. Zirnbauer and F.D.M. Haldane, Phys. Rev. B **52**, 8729 (1995).

[12] M. Pustilnik *et al.*, Phys. Rev. Lett. **91**, 126805 (2003).

[13] B.D. Simons, P.A. Lee, and B.L. Altshuler, Phys. Rev. Lett. **70**, 4122 (1993); Phys. Rev. B **48**, 11450 (1993); Nucl. Phys. B **409**, 487 (1993).

[14] At $\lambda = 1/2, 1, 2$ the structure factor of CSM maps to the parametric correlations in Random Matrix Theory [13].

[15] The square-root divergencies in $S(q,\omega)$ at $\omega \to \omega_\mp$ for $\lambda = 2^{\pm 1}$ have been noticed in E.R. Mucciolo *et al.*, Phys. Rev. B **49**, 15197 (1994).

[16] A.P. Polychronakos, in *Topological Aspects of Low Dimensional Systems*, eds. A. Comtet *et al.* (Springer,

2000), pp. 415-472.
[17] G.D. Mahan, *Many-Particle Physics* (Springer, 2000).
[18] T. Ogawa, A. Furusaki, and N. Nagaosa, Phys. Rev. Lett. **68**, 3638 (1992); L. Balents, Phys. Rev. B **61**, 4429 (2000).
[19] The Fourier transform of Eq. (4) is easily found with the help of the identity $(\sin\alpha)^{-2} = -2\sum_n |n| e^{2in\alpha}$ and reads $V_q = -\lambda(\lambda-1)\pi|q|/m$.
[20] M. Pustilnik, in progress.